On Randomly Weighted, Randomly Perturbed Dense Graphs

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We let G_0 be a dense graph or digraph and we add random edges R to create a graph or digraph $G = G_0 + R$.
Introduced by Bohman,F,Martin in 2003.
Many recent papers.

Anastos, F in 2019 randomly colored the edges and considered (i) rainbow Hamilton cycles and (ii) rainbow connectivity.

Aigner-Horev and Hefetz 2020 gave improvements to (i) and Balogh,Finlay and Palmer 2021 gave improvements to (ii).

On Randomly Weighted, Randomly Perturbed Dense Graphs

In this talk we also give the edges of G independent random weights.
Randomly adding randomly weighted, randomly colored edges, not yet considered.

On Randomly Weighted, Randomly Perturbed Dense Graphs

In this talk we also give the edges of G independent random weights.

We consider the case where G_0 is αn-regular.

1. Minimum Spanning Trees.
2. Shortest Paths.
3. Perfect matchings in bipartite graphs.
4. Asymmetric TSP (with Peleg Michaeli).

Minimum Spanning Trees

Let G be an asymptotically r-regular and suppose each edge has an independent uniform $[0,1]$ random weight.

Let L_n be the (random) minimum weight of a spanning tree.

Theorem (Beveridge,F,McDiarmid)

Suppose that $r \to \infty$. Given a moderate connectivity condition MCC, $L_n \sim \frac{n}{r}\zeta(3)$.

1. $G = K_n : \; L_n \sim \zeta(3)$.
2. $G = K_{n,n} : \; L_n \sim 2\zeta(3)$.
3. $G = Q_n : \; L_n \sim \frac{2^n}{n}\zeta(3)$.

Adding $o(n^2)$ randomly weighted random edges to an asymptotically αn-regular graph satisfies MCC w.h.p.
So, w.h.p. $L_n \sim \alpha^{-1}\zeta(3)$.

Shortest Paths

The edges of $G_0 + R$ are given independent $EXP(1)$ random lengths – $\mathbf{Pr}(EXP(1) \geq \lambda) = e^{-\lambda}$.
Let $d_{i,j}$ denote the length of a shortest path from i to j.

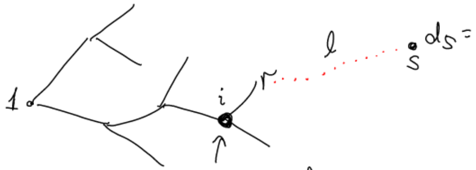
Theorem

1. $d_{1,2} \sim \frac{\log n}{\alpha n}$ *w.h.p.*
2. $\max_j d_{1,j} \sim \frac{2 \log n}{\alpha n}$ *w.h.p.*
3. $\max_{i,j} d_{i,j} \sim \frac{3 \log n}{\alpha n}$ *w.h.p.*

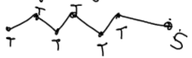
Janson 1999 proved this when $\alpha = 1, R = \emptyset$.

We adapt his argument.

Dijkstra Algorithm



d_s = length of shortest path



$d_r + \ell \geq d_k$

$\Rightarrow \ell = d_k - d_r + E(1)$

d_i = length of shortest path 1 to i

T

k vertices

$0 = d_1 \leq d_2 \leq \cdots \leq d_k$

ν_k edges from T to $\overline{T}$

$\Rightarrow d_{k+1} - d_k = \min \nu_k$ indep. $E(1)$'s

$E(d_{k+1} - d_k) = \dfrac{1}{\nu_k}$

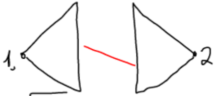
Shortest Paths

So $E(d_m) = \frac{1}{\nu_1} + \frac{1}{\nu_2} + \cdots + \frac{1}{\nu_m}$

Complete Graph: $\nu_k = k(n-k)$

H+R: $\nu_k \approx \alpha k n$ for $k = o(n)$

$d_{1,2}$: Upper Bound



$m = \sqrt{\omega n}$ vertices in each tree

Cheap random edge

$$2 \sum_{k=1}^{\sqrt{\omega n}} \frac{1}{\alpha k n} \approx \frac{\log n}{\alpha n}$$

$+ o(\log n / n)$

Lower Bound Similar

Perfect Matchings

Perfect matchings in $K_{n,n}$: M_n is the minimum weight matching.

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5. Parisi 1998: Conjecture $\mathbf{E}(w(M_n)) = \sum_{i=1}^{n} \frac{1}{i^2}$.

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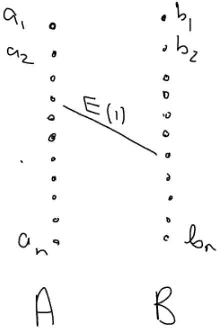
5. Parisi 1998: Conjecture $\mathbf{E}(w(M_n)) = \sum_{i=1}^{n} \frac{1}{i^2}$.
6. Linusson,Wastlund 2004 and Nair,Prabhakar,Sharma 2005 verified the Parisi conjecture.

Assume edge weights are exponential mean 1.



M_r = min. wt matching of $A_r=\{a_1,a_2,\dots,a_r\}$ into B

Parisi conjecture equivalent to

$$E[w(M_r)-w(M_{r-1})]=\frac{1}{r}\left[\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{n-r+1}\right]$$

Lovely "proof" next.

Perfect Matchings – Wästlund

$X = w(M_r)$ and Y_i is the minimum weight of a matching from $A_r \setminus \{a_i\}$ into B.

Add a new vertex b^* and edges of length $EXP(\lambda)$ from A_r to $B^* = B \cup \{b^*\}$.

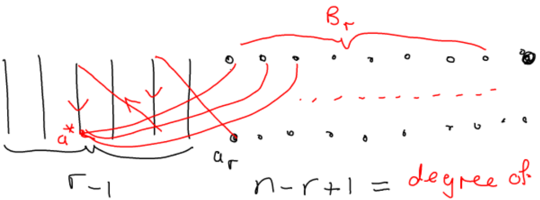
$$\begin{aligned}
P(r) = \mathbf{Pr}(b^* \in M_r^*) &= \mathbf{E}\left(\sum_{i=1}^{r} \mathbf{Pr}(w(a_i, b^*) < X - Y_i)\right) \\
&= \mathbf{E}\left(\sum_{i=1}^{r} (1 - e^{-\lambda(X - Y_i)})\right) \\
&= \lambda \sum_{i=1}^{r} \mathbf{E}(X - Y_i) + O(\lambda^2) \\
&= \lambda r(\mathbf{E}(w(M_r)) - \mathbf{E}(w(M_{r-1})) + O(\lambda^2)).
\end{aligned}$$

Perfect Matchings – Wästlund

$\underline{\underline{H+R}}$

$$\Pr(b^* \in M_r^* \mid b^* \notin M_{r-1}^*) = \frac{\lambda}{n-r+1+\lambda}$$

$K_{n,n}$



$n-r+1 =$ degree of a^* in B_r.

a^* = end of augmenting path.

$$P(r) = \lambda r(\mathbf{E}(w(M_r)) - \mathbf{E}(w(M_{r-1})) + O(\lambda^2))$$
$$= \mathbf{Pr}(b^* \in M_r^*) = 1 - \prod_{i=0}^{r-1} \mathbf{Pr}(b^* \in M_{r-i}^* \mid b^* \notin M_{r-i-1}^*)$$

In $K_{n,n}$ we have

$$P(r) = 1 - \prod_{i=0}^{r-1} \frac{n-i}{n-i+\lambda} = \lambda \sum_{i=0}^{r-1} \frac{1}{n-i} + O(\lambda^2).$$

In $G_0 + R$ we have

$$P(r) = \lambda \sum_{i=0}^{r-1} \frac{1}{\delta_i} + O(\lambda^2)$$

where δ_r is the degree of a^* in B_r.

Perfect Matchings

We should have $\delta_r \sim \alpha(n-r+1)$ implying that

$$\mathbf{E}(w(M_n)) \sim \frac{\pi^2}{6\alpha}.$$

We can only prove this when G_0 is pseudo-random in the sense of Thomason 1989.

The proof is a bit technical.

Asymmetric TSP

Let $C(i,j)$ be independent $EXP(1)$ for $1 \leq i \neq j \leq n$.

We let the $C(i,j)$ be the costs for an instance of the Asymmetric Travelling Salesperson Problem (ATSP). I.e. the minimum cost of a Hamilton cycle in the complete digraph $\vec{K}_n$.

The Assignment Problem (AP) is the problem of finding the minimum cost perfect matching in $K_{n,n}$ with costs $C(i,j)$.

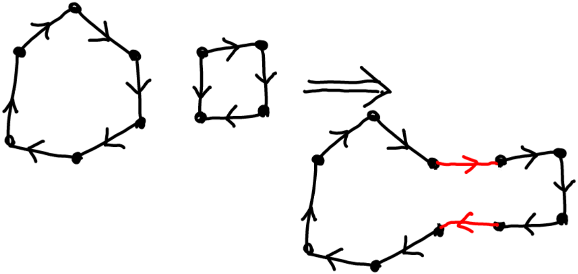
AP is equivalent to finding the minimum cost of a collection of vertex disjoint cycles in $K_{n,n}$ that cover all vertices.

It follows that in terms of optimal values $val(AP) \leq val(ATSP)$.

Asymmetric TSP

Karp 1979 descibed a *patching* algorithm that in $O(n^3)$ time w.h.p. produces a Hamilton cycle H_{karp} of weight $(1+o(1))val(AP) \leq (1+o(1))val(ATSP)$.

The algorithm solves AP and then patches the associated cycles together as cheaply as possible.



Asymmetric TSP

We replace $\vec{K}_n$ with $D = D_0 + R$ where D_0 has minimum in- and out-degree at least αn.

Theorem

Suppose that $|R| = n^{2-\epsilon}$ *and that each edge of* D *is given an independent* $EXP(1)$ *cost. Then w.h.p.* $val(ATSP) = (1 + o(1))val(AP)$ *and Karp's patching algorithm finds a tour of the claimed cost in polynomial time.*

The proof rests on the following lemma:

Lemma

(a) W.h.p., the solution to AP contains only edges of cost $C(i,j) \leq \gamma_n = n^{-(1-2\epsilon)}$.
(b) W.h.p., after solving AP, the number ν_C *of cycles is at most* $r_0 \log n$ *where* $r_0 = n^{1-3\epsilon}$.

Asymmetric TSP

Given the lemma, the proof is simple. Let $\mathcal{C} = C_1, C_2, \ldots, C_\ell$ be a cycle cover and let $k_i = |C_i|$ where $k_1 \leq k_2 \leq \cdots \leq k_\ell$, $2 \leq \ell \leq r_0$. Different edges in C_i give rise to disjoint patching pairs. We only consider the random edges R when looking for a patch. The number of possible patching pairs $\nu_{\mathcal{C}}$ satisfies

$$\nu_{\mathcal{C}} \geq \sum_{i \neq j} k_i k_j = \frac{1}{2}\left(n^2 - \sum_{i=1}^{\ell} k_i^2\right)$$
$$\geq \frac{1}{2}\left(n^2 - ((n-\ell+1)^2 + \ell - 1)\right) \geq \frac{\ell n}{2}.$$

Each of these $\nu_{\mathcal{C}}$ pairs uses a disjoint set of edges.

We define the sets

$$R_\ell = \left\{ e \in R : C(e) \le \gamma_n + \left(\frac{\log n}{\ell n^{1-5\epsilon/2}} \right)^{1/2} \right\},\ 1 \le \ell \le r_0.$$

Let $\mathcal{E}_\ell$ be the event that $|\mathcal{C}| = \ell$ and that there is no patch using only edges in R_ℓ. If $\mathcal{E}_\ell$ does not occur then we reduce the number of cycles by at least one. We have

$$\mathbf{Pr}(\mathcal{E}_\ell) \le \exp\left\{ -\frac{n^{4-2\epsilon}\log n}{18 n^{4-5\epsilon/2}} \right\} = o(n^{-1}).$$

It follows that $\mathbf{Pr}(\exists i : \mathcal{E}_i) = o(1)$ and then the union bound implies that w.h.p. the patches involved in these cases add at most the following to the cost of the assignment:

$$\sum_{\ell=1}^{r_0 \log n} \left(\gamma_n + \left(\frac{\log n}{\ell n^{1-5\epsilon/2}} \right)^{1/2} \right) \le r_0 \gamma_n \log n + \left(\frac{2 r_0 \log^2 n}{n^{1-5\epsilon/2}} \right)^{1/2} = o(1).$$

THANK YOU