Ramsey-type Results for Randomly Perturbed Graphs

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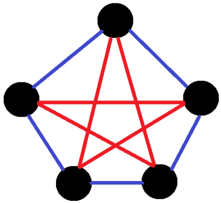
Joint work with Shagnik Das

Let $H_1, \dots, H_r$ be graphs.

- A graph G is $(H_1, \dots, H_r)$-Ramsey if whenever the edges of G are r-coloured, there is a monochromatic copy of H_i in colour i in G *for some* $i \in [r]$.

Let H be a graph and $r \in \mathbb{N}$.

- A graph G is (H, r)-Ramsey if whenever the edges of G are r-coloured, there is a monochromatic copy of H in G.



K_5 is not $(K_3, 2)$-Ramsey

Ramsey properties of random graphs

The random graph $G(n, p)$ has:

- Vertex set $[n] := \{1, \dots, n\}$;
- Each edge is present with probability p, independently of all other choices.

Question

For which values of p is $G(n, p)$ a.a.s. (H, r)-Ramsey?

Ramsey properties of random graphs

Given a graph H define

$$m_2(H) := \max\left\{\frac{e(H')-1}{v(H')-2} \,:\, H' \subseteq H \text{ and } v(H') \geq 3\right\}.$$

Theorem (Rödl and Ruciński 1995)

- *Suppose H is not a forest consisting of stars or paths of length 3;*
- $r \geq 2$.

Then there exist $c, C > 0$ s.t.

$$\lim_{n\to\infty} \mathbb{P}[G(n,p) \text{ is } (H,r)\text{-Ramsey}] = \begin{cases} 0 & \text{if } p < cn^{-1/m_2(H)}; \\ 1 & \text{if } p > Cn^{-1/m_2(H)}. \end{cases}$$

Given $t \geq s \geq 3$,

$$m_2(K_t, K_s) := \frac{\binom{t}{2}}{t - 2 + 1/m_2(K_s)}$$

Theorem (Marciniszyn, Skokan, Spöhel and Steger + Balogh, Morris and Samotij)

There exist $c, C > 0$ s.t.

$$\lim_{n \to \infty} \mathbb{P}[G(n,p) \textit{ is } (K_t, K_s)\text{-}\textit{Ramsey}] = \begin{cases} 0 & \textit{if } p < cn^{-1/m_2(K_t,K_s)}; \\ 1 & \textit{if } p > Cn^{-1/m_2(K_t,K_s)}. \end{cases}$$

- Krivelevich, Sudakov and Tetali raised this problem in 2006

Theorem (Krivelevich, Sudakov and Tetali 2006)

$p = n^{-2/(t-1)}$ is the threshold for the (K_t, K_3)-Ramsey property in randomly perturbed graphs (for all $t \geq 3$).
That is,

- *if $p = \omega(n^{-2/(t-1)})$ then, given any n-vertex graph G of density $d > 0$, a.a.s. $G \cup G(n,p)$ is (K_t, K_3)-Ramsey;*
- *there is an n-vertex graph G of 'positive density' such that a.a.s. $G \cup G(n,p)$ is not (K_t, K_3)-Ramsey if $p = o(n^{-2/(t-1)})$.*

Theorem (Das and T. 2020)

Let $t \geq s \geq 5$. $p = n^{-1/m_2(K_t, K_{\lceil s/2 \rceil})}$ is the threshold for the (K_t, K_s)-Ramsey property in randomly perturbed graphs

Powierski independently proved the case of $s = t \geq 5$ odd

Ramsey properties of randomly perturbed graphs

- The (K_t, K_s)-Ramsey problem resolved for $t \geq s$ unless $s = 4$
- 'Something else' going on in these cases.

More precisely:

- If $s = 4$, then $G(n, p)$ is $(K_t, K_{\lceil s/2 \rceil})$-Ramsey precisely if $G(n, p)$ contains a copy of K_t. Only need $p = \omega(n^{-2/(t-1)})$ for this.
- Das-T.: constructions show that this isn't the correct threshold for the (K_t, K_4)-Ramsey problem.
- Further, a construction of Powierski together with a proof of Das-T. shows that $p = n^{-1/2}$ is the threshold for a randomly perturbed graph to be (K_4, K_4)-Ramsey

Open Problem

Resolve (K_t, K_4)-Ramsey problem for $t \geq 5$

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Further discussion: the general problem

- As our main result illustrates, often the threshold for a Ramsey property in randomly perturbed graphs is tied to an asymmetric Ramsey property in random graphs
- The Kohayakawa–Kreuter conjecture, would, if true, resolve this latter problem
- 1-statement resolved by Mousset, Nenadov and Samotij (2020)
- 0-statement progress due to: Kohayakawa and Kreuter (1997); Marciniszyn, Skokan, Spöhel and Steger (2009); Liebenau, Mattos, Mendonca and Skokan (2020+); Hyde (2021+).