Schur's Theorem for randomly perturbed sets

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Randomly Perturbed (Hyper)graphs

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Introduction

An origin story

Theorem (Dirac, 1952)

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The random graph $G(n, p)$ is Hamiltonian with probability tending to

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Theorem (Bohman–Frieze–Martin, 2003)

If $\delta(G) = \Omega(n)$, then adding $\Theta(n)$ random edges makes G Hamiltonian with high probability.

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We say a set $A \subseteq \mathbb{N}$ is r-Schur if every r-colouring of A has a monochromatic Schur triple.

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How large can a subset $A \subseteq [n]$ be without being r-Schur?

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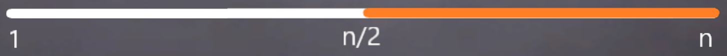

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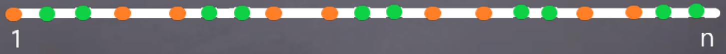

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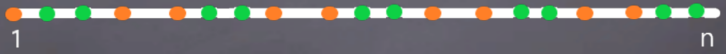

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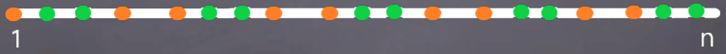

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- Conjecture from Abbott and Wang (1977)

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For $r = 2$, the following hold.

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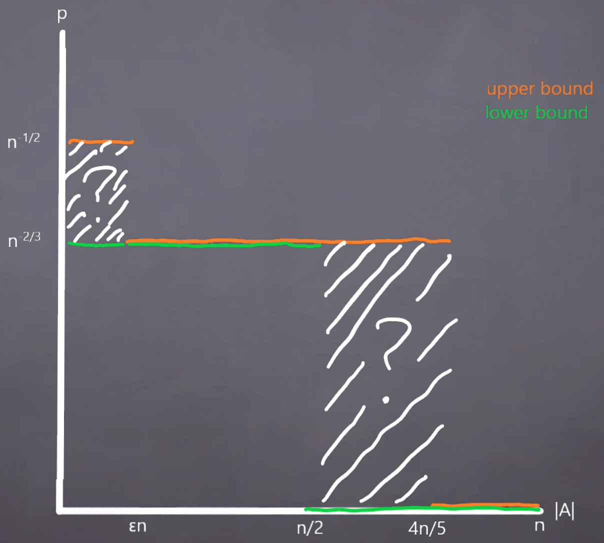
Result is best possible for $|A| \leq \frac{n}{2}$: take A sum-free, and $A \cup [n]_p$

A pictorial summary



Our results

Denser sets

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For $n, t \in \mathbb{N}$ with $\frac{n}{2} + t \leq \frac{4n}{5}$, define $p(n, t) = \min\left\{n^{-2/3}, t^{-1}\right\}$.

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Observation

We need $t = \omega\left(n^{2/3}\right)$ *before we save any further randomness.*

Perturbed stability

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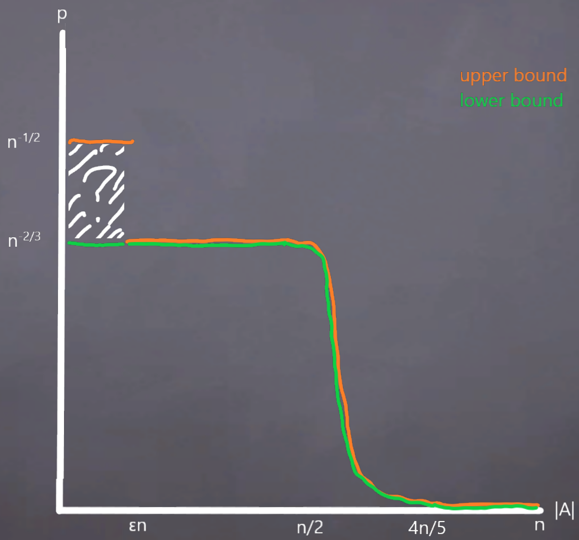
  - $\Rightarrow$ A_{large} *"more sum-free" than* A_{odd}

An updated picture



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Question

What happens when our initial set is moderately sparse, $\Omega\left(n^{1/2}\right) = |A| = o(n)$?

Sparse results

Theorem (D.–Knierim–Morris, 2022+)

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Remark

- *Lower bound interpolates between* $n^{-1/2}$ *and* $n^{-2/3}$
- *Upper bound is* $o\left(n^{-1/2}\right)$ *when* $s = \tilde{\omega}\left(n^{1/2}\right)$

Proof sketches

Lower bound

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  - Such a cycle unlikely to exist in random elements $\square$

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Theorem (D.–Knierim–Morris, 2022+)

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  - ▶ apply union bound more efficiently $\square$

The colouring hypergraph — vertices

Definition (Colouring hypergraph)

Given $A \subseteq [n]$, $|A| = s$, the colouring hypergraph $\mathcal{H}_A$ has vertices $V(\mathcal{H}_A) = V_R \cup V_G$, which are two disjoint copies of $[n]$.

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Given any $S \subseteq [n]$, map colourings $\varphi : S \to \{\text{red}, \text{green}\}$ to

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For every $a \in A$ *and* $x, y, z, w \in [n]$ *such that* a, x, y *and* a, z, w *form Schur triples, we add a 4-edge on the vertices* $x, y \in V_R$ *and* $z, w \in V_G$.

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$\Rightarrow$ *Any Schur colouring of $A \cup [n]_p$ is independent in $\mathcal{H}_A$*

Containers for colourings

Proposition

For every $\varepsilon > 0$ there is some $c = c_\varepsilon$ such that, if $A \subseteq [n]$ is of size $s = \Omega(n^{1/2})$, then there is a collection $\mathcal{C}$ of subsets of $V(\mathcal{H}_A)$ for which:

1. *For every $P \subseteq [n]$ and Schur colouring φ of $A \cup P$, there is some $C \in \mathcal{C}$ such that $\varphi \subseteq C$.*
2. *For every $C \in \mathcal{C}$, $e(\mathcal{H}_A[C]) \le \varepsilon s n^2$.*
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Goal

Show that for every container $C \in \mathcal{C}$, the probability that $P \sim [n]_p$ admits a Schur colouring φ of $A \cup P$ with $\varphi \subseteq C$ is very small.

Anatomy of a container

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Given $C \subseteq V(\mathcal{H}_A)$, we can partition $[n]$ into four sets:

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Observation

- *C prescribes the colours of the elements in R_C and G_C*
- *Elements from M_C cannot be coloured*

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There are three types of containers in the world

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- Almost all elements in $[n]$ receive at least one colour
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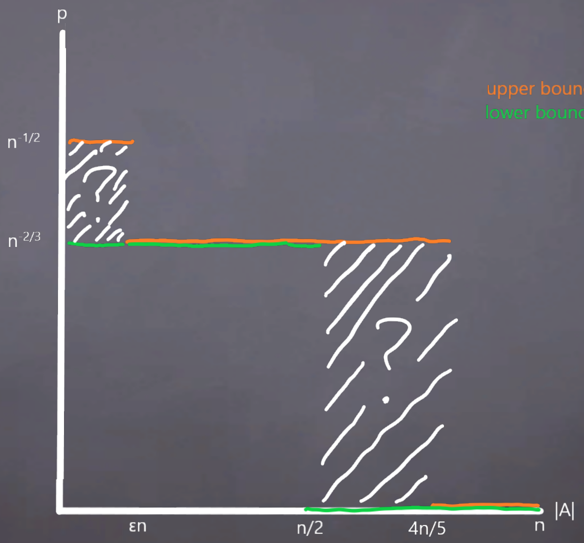
Concluding remarks

Progress made



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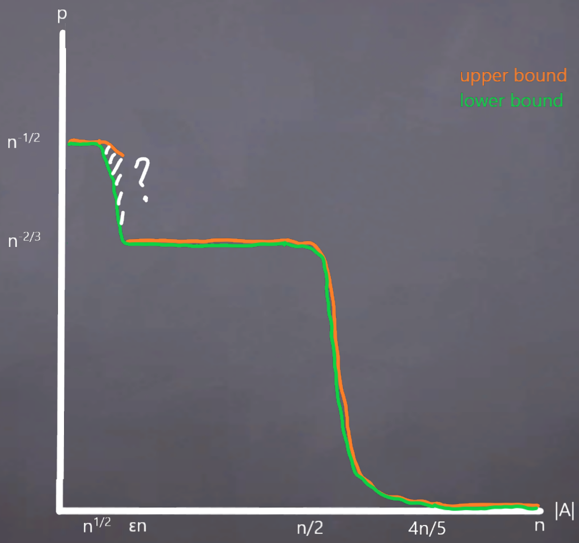

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Thank you for
your attention!