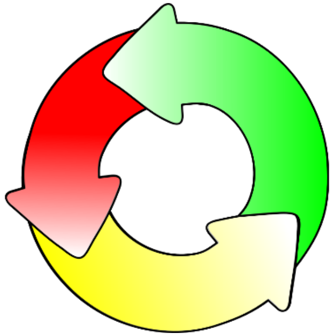
Cycle lengths in randomly perturbed graphs



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Notation

$G = (V, E)$ – graph, $|V| = n$

$L(G) \coloneqq$ set of cycle lengths in G

G is pancyclic if $L(G) = \{3, \dots, n\}$

G is Hamiltonian if contains a Hamilon cycle, i.e, $n \in L(G)$

A general meta-question…

$G = (V, E)$, $|V| = n$

Assume: $\alpha(G) \leq t$ $\left(\Rightarrow \chi(G) \geq \frac{n}{t}\right)$

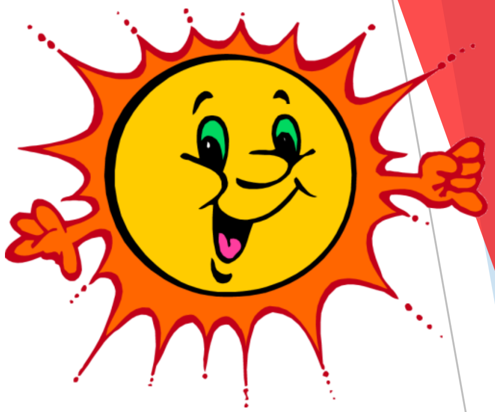
and/or $\delta(G) \geq k$

? cycle lengths in G?

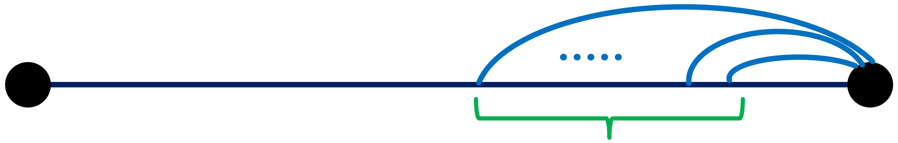
? structure of $L(G)$?

? long cycles in $L(G)$?

On the bright side of it…



- Trivial: $\delta(G) \geq k \Rightarrow \geq k-1$ distinct cycle lengths



- Gao, Huo, Liu, Ma'22

  (conjecture of Liu,Ma'18; $k = 3$ – Bondy, Vince'98)

  $\delta(G) \geq k \Rightarrow L(G)$ contains an AP of length $k-1$,

  difference $\in \{1,2\}$

- GHLM'22 (conjecture of Sudakov, Verstraete'17) :

  $\chi(G) \geq k \Rightarrow L(G)$ contains cycles of $k-2$ consecutive lengths

…

Not every day is sunny in Pittsburgh…

(or even in Tel Aviv…)

Ex.: $G =$ collection of t disjoint cliques of size $\frac{n}{t}$ each

- $\alpha(G) = t$
- $\delta(G) \approx \frac{n}{t}$
- no cycles longer than $\frac{n}{t}$; $|L(G)| < \frac{n}{t}$

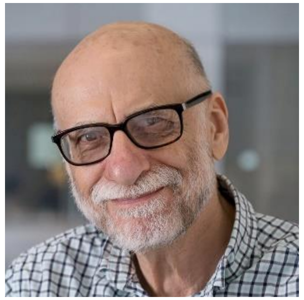
$\Rightarrow$ need some fix…

Little randomness comes to the rescue…

Possible remedy: add few <u>random</u> edges on top of G

$\Rightarrow$ <u>randomly perturbed graphs</u>

Bohman, Frieze, Martin’03:



<u>Th. A</u>: $G = (V, E)$, $|V| = n$, $\delta(G) \geq \delta n$, $\delta \in (0,1)$ – constant

$+\, C(\delta)n$ random edges $=: R$

$\Rightarrow$ <u>whp</u> $G \cup R$ is Hamiltonian

(in fact even pancyclic)

Little randomness comes to the rescue… (cont.)

Ex.: $G = K_{\frac{n}{3}, \frac{2n}{3}}$, sides $A, B, |A| = \frac{n}{3}, |B| = \frac{2n}{3}$

$\forall G' \supseteq G$, if C is a Hamilton cycle in G'

$\Rightarrow C$ has $\geq \frac{n}{3}$ edges inside B

$\Rightarrow |E(G')| - |E(G)| \geq \frac{n}{3}$ – deterministically

Too much independence can hurt you…

Perhaps: large $\alpha(G)$ requires many random edges?

Th. B (BFM'03): $G = (V, E)$, $|V| = n$, $\delta(G) \geq \delta n$, $\delta = \delta(n)$

$$\boxed{\alpha(G) < \frac{\delta^2 n}{2}}$$

$$R \sim G(n,p),\;\; p \gg \frac{\log(\frac{1}{\delta})}{\delta^3 n^2}$$

$\Rightarrow$ whp $G \cup R$ is Hamiltonian

? Whether the bound for $p(n)$ is sharp?

Observe: $\delta = O(n^{-\frac{1}{3}}) \Rightarrow$ require $p \gg \frac{\log n}{n}$

— higher than the threshold for Ham'ty in $G(n,p)$

$\Rightarrow$ no need in the background graph G…

Our results

Th. 1: $G = (V, E)$, $|V| = n$, $\delta(G) \geq \delta n$,

$$\Omega\left(n^{-\frac{1}{3}}\right) = \delta = o(1)$$

$$\alpha(G) \leq c\delta^2 n$$

$$p \geq \frac{C \log(\frac{1}{\delta})}{\delta n^2}, \qquad R \sim G(n,p)$$

$\Rightarrow$ whp $G \cup R$ is pancyclic

Tightness: $G = \frac{1}{\delta}$ disjoint cliques of size $\approx \delta n$

$$\delta = \Omega\left(n^{-\frac{1}{3}}\right); \qquad \alpha(G) = \frac{1}{\delta} \leq c\delta^2 n$$

Easy to see: need $\geq \frac{C \log(\frac{1}{\delta})}{\delta}$ random edges to touch every clique whp

Theorem 1 – proof idea

A very useful tool:

Lemma (BFKM'04): $G = (V, E)$, $|V| = n$, $\delta(G) \geq k$

Can decompose: $V = V_1 \dot{\cup} \dots \dot{\cup} V_t$ s.t.

- $|V_i| \geq \frac{k}{8}$, $\;i = 1, \dots, t$
- $G[V_i]$ is $\frac{k^2}{16n}$ – connected

Illustration: BFM for the dense case $\delta(G) = k = \Theta(n)$

Assume: $\alpha(G) < \frac{k^2}{16n}$

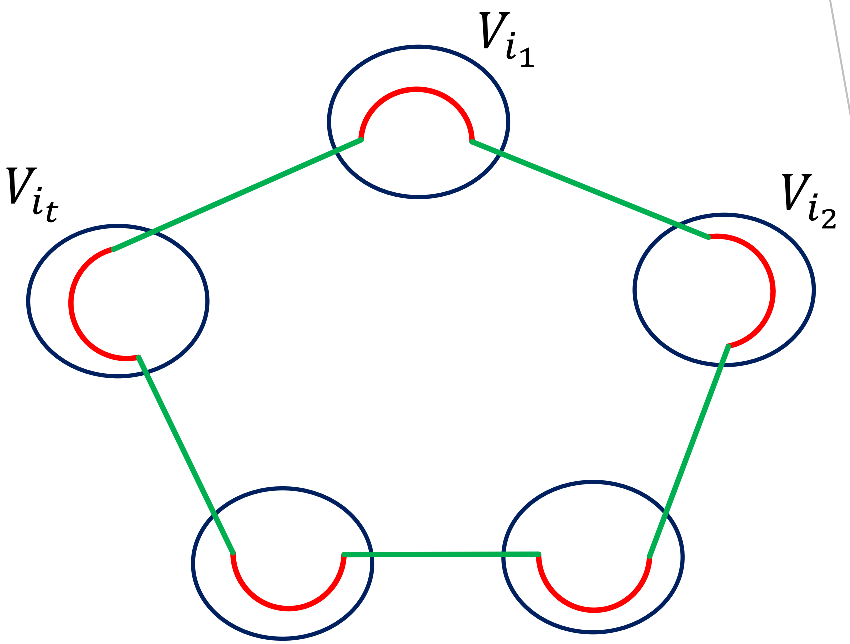
Decompose into $t = O(1)$ pieces; $\kappa(G[V_i]) \geq \frac{k^2}{16n} > \alpha(G) \geq \alpha(G[V_i])$

$\Rightarrow G[V_i]$ is Hamilton-connected (Chvátal-Erdős'72)

Theorem 1 – proof idea (cont.)

- use $\omega(1)$ random edges from R
  to weave the pieces V_i
  into a Hamilton cycle

- use Ham. connectivity of $G[V_i]$
  to connect entry/exit points inside V_i
  by a Hamilton path in $G[V_i]$



Get a Hamilton cycle in $G \cup R$ whp

Theorem 1 — proof idea (cont.)

General case $\delta(G) = o(n)$:

- same decomposition
- more sophisticated weaving:

  HC in the auxiliary graph — enters pieces V_i possibly many times

  (pieces of different sizes)

⇒ need a statement:

∃ family of paths connecting given pairs

in a highly connected graph

and covering all vertices

(variant of linkage)

— prove such a statement using randomness.

Bounding $\alpha(G)$ only

Th. 2: $G = (V, E)$, $|V| = n$, $\omega(1) = \alpha(G) \leq cn$

$\epsilon > 0$

$p \geq \frac{C\alpha(G)}{n^2}$, $R \sim G(n, p)$

$\Rightarrow$ whp $G \cup R$ contains a cycle $C, |C| \geq (1 - \epsilon)n$

(nearly Hamilton)

Tightness: $G = t$ disjoint cliques of size $\frac{n}{t}$

$\forall\, R, |E(R)| < ct$

circumference$(G \cup R) \leq cn$

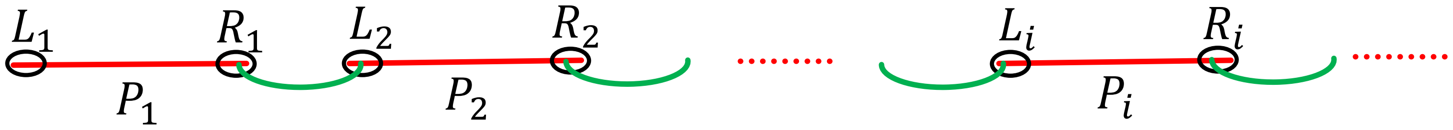
Theorem 2 – proof idea

- $\alpha(G) \le t \;\Rightarrow$ most vertices $v \in V, d(v) = \Omega\left(\frac{n}{t}\right)$

  $\Rightarrow$ can repeatedly find (in G) disjoint paths $P_1, \dots, P_s, |P_i| = \Theta\left(\frac{n}{t}\right)$

  $\bigcup_{i=1}^{s} P_i$ covers $\ge \left(1 - \frac{\epsilon}{2}\right) n$ vertices

- use random edges from R to weave (most of) these paths:



$\Rightarrow$ whp get a path/cycle of length $\ge (1 - \epsilon)n$.

Getting tough…

Def.: $G = (V, E)$ is t –tough if $\forall S \subset V$

$\# \text{ conn. comps of } G - S \leq \max\left\{1, \frac{|S|}{t}\right\}$

Chvátal'73:

- observed: G – Hamiltonian $\Rightarrow$ G is 1-tough
- conjectured:

Toughness conjecture: $\exists\, t \geq 1$ constant s.t.

every t-tough graph is Hamiltonian

— still widely open.

Our result

- Assuming toughness, one of several regimes:

Th. 3: Assume toughness conjecture with constant t_0

$G = (V, E)$, $|V| = n$,

$\delta(G) = k = O(\sqrt{n})$

$\alpha(G) \leq c(t_0)k$

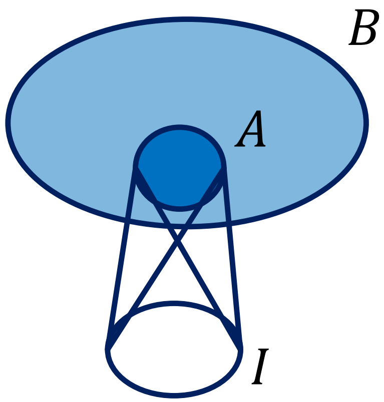
$p = \frac{C \log n}{nk}$, $\quad R \sim G(n, p)$

$\Rightarrow$ whp $G \cup R$ is pancyclic

Our result (cont.)

Th. 3: Assume toughness conjecture with constant t_0
$G = (V, E)$, $|V| = n$,
$\delta(G) = k = O(\sqrt{n})$
$\alpha(G) \le c(t_0)k$
$p = \frac{C \log n}{nk}$, $R \sim G(n, p)$
$\Rightarrow$ whp $G \cup R$ is pancyclic

Tightness:

$G =$ 

$G[B]$ – clique
$A \subset B, |A| = \frac{k}{2}$

I – indep. set, $|I| = k - 1$
I fully connected to A

$\delta(G) = \frac{k}{2}$

$\alpha(G) = k$

Need: $\ge \frac{k}{2}$ edges of R touching I to get a 1-tough graph

$\Rightarrow$ require: $|R| = \Omega(n)$


That's all Folks!