Clique-factors in Randomly Perturbed Hypergraphs

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Based on joint works with
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Clique-factors: Hajnal–Szemerédi Theorem

- Simple, dense graphs G. $\delta(G)$: the minimum degree of G
- Hajnal-Szemerédi '70, Corrádi-Hajnal '63: n-vertex graph G with $\delta(G) \geq (1 - 1/r)n$ contains K_r-factor.
  - Generalized to F-factors by Alon-Yuster '00, Kühn-Osthus '09
  - Algorithmic version: Hell-Kirkpatrick '83, H.-Treglown '20
  - Hypergraph extension still unknown

For k-uniform Hypergraphs H

- For $1 \leq d < k$, $\delta_d(H) := \min\{\deg(S) : |S| = d\}$.
- Keevash-Mycroft '11: n-vertex 3-graph H with $\delta_2(H) \geq 3n/4$ contains K_4^3 -factor.
  - The only known tight result on cliques besides those on matchings
  - A long and involved proof using Hypergraph Regularity Lemmas and Hypergraph Blow-up Lemma
  - A simpler proof found by Han. [2021]

Clique-factors in random graphs

- $G(n,p)$: n-vertex graph, where each pair of vtxs form an edge with prob=p.
- Posa '76, Korshunov '77: $p \geq \frac{\log n}{n} \Rightarrow$ Hamiltonian cycle in $G(n,p)$ whp.
- Johansson-Kahn-Vu '08:
  - $p = \omega(n^{-2/r}(\log n)^{2/(r^2-r)}) \Rightarrow$ whp. $G(n,p)$ has a K_r-factor.
  - $p = o(n^{-2/r}(\log n)^{2/(r^2-r)}) \Rightarrow$ whp. $G(n,p)$ has no K_r-factor.

Randomly perturbed Model

- "Adding random edges to (dense) deterministic (hyper)graphs decreases the min-degree / density requirements"
- "Adding (a small number of) random edges to (dense) deterministic (hyper)graphs decreases the min-degree / density requirements"
- Bohman-Frieze-Martin '03: Suppose G is a graph with $\delta(G) \geq \alpha n$. Add Cn uniformly random edges to G. Then the resulting graph whp. is Hamiltonian.
- In other words, $G \cup G(n, C/n)$ is Hamiltonian whp.

Clique-factors in randomly perturbed graphs

- Johansson-Kahn-Vu ’08:
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  - $p = o(n^{-2/r}(\log n)^{2/(r^2-r)}) \Rightarrow$ whp. $G(n,p)$ has no K_r-factor.
- Balogh-Treglown-Wagner ’19: Suppose $\mathcal{G}_\alpha = \{G : \delta(G) \geq \alpha|G|\}$
  - $p = \omega(n^{-2/r}) \Rightarrow \forall G \in \mathcal{G}_\alpha$ whp. $G \cup G(n,p)$ has a K_r-factor.
  - $p = o(n^{-2/r}) \Rightarrow$ there exists $G' \in \mathcal{G}_\alpha$ whp. $G' \cup G(n,p)$ has no K_r-factor.

Clique-factors in randomly perturbed graphs

- Balogh-Treglown-Wagner is tight for small $\alpha \in (0,1/r)$.
- H.-Morris-Treglown, '21 determined the optimal p for almost all α:
  - For $2 \leq k \leq r$ and $1 - \frac{k}{r} < \alpha < 1 - \frac{k-1}{r}$,
    - $p = \omega(n^{-2/k}) \Rightarrow \forall G \in \mathcal{G}_\alpha$ whp. $G \cup G(n,p)$ has a K_r-factor.
    - $p = o(n^{-2/k}) \Rightarrow$ there exists $G' \in \mathcal{G}_\alpha$ whp. $G' \cup G(n,p)$ has no K_r-factor.
- E.g. $r=4$. p: $n^{-\frac{1}{2}}$ $n^{-\frac{2}{3}}$ n^{-1} 0

  α: 0 $\frac{1}{4}$ $\frac{1}{2}$ $\frac{3}{4}$ 1

Perturbed k-graphs

- Krivelevich-Kwan-Sudakov '16, for $k \geq 3$, $\alpha > 0$,
  - $p = \omega(n^{1-k}) \Rightarrow \forall H$ with $\delta_{k-1}(H) \geq \alpha n$, whp. $H \cup H^{(k)}(n,p)$ has a perfect matching.
- They also raised the analogous question for weaker minimum degree conditions.
- Chang-H.-Kohayakawa-Morris-Mota, '21. Their result holds for all H satisfying $\delta_1(H) \geq \alpha n^{k-1}$.
  - A general result for F-factors was obtained, tight e.g. when F is k-partite, $F = K_4^{3-}$, or F is the Fano plane.

Perturbed k-graphs: clique-factors?

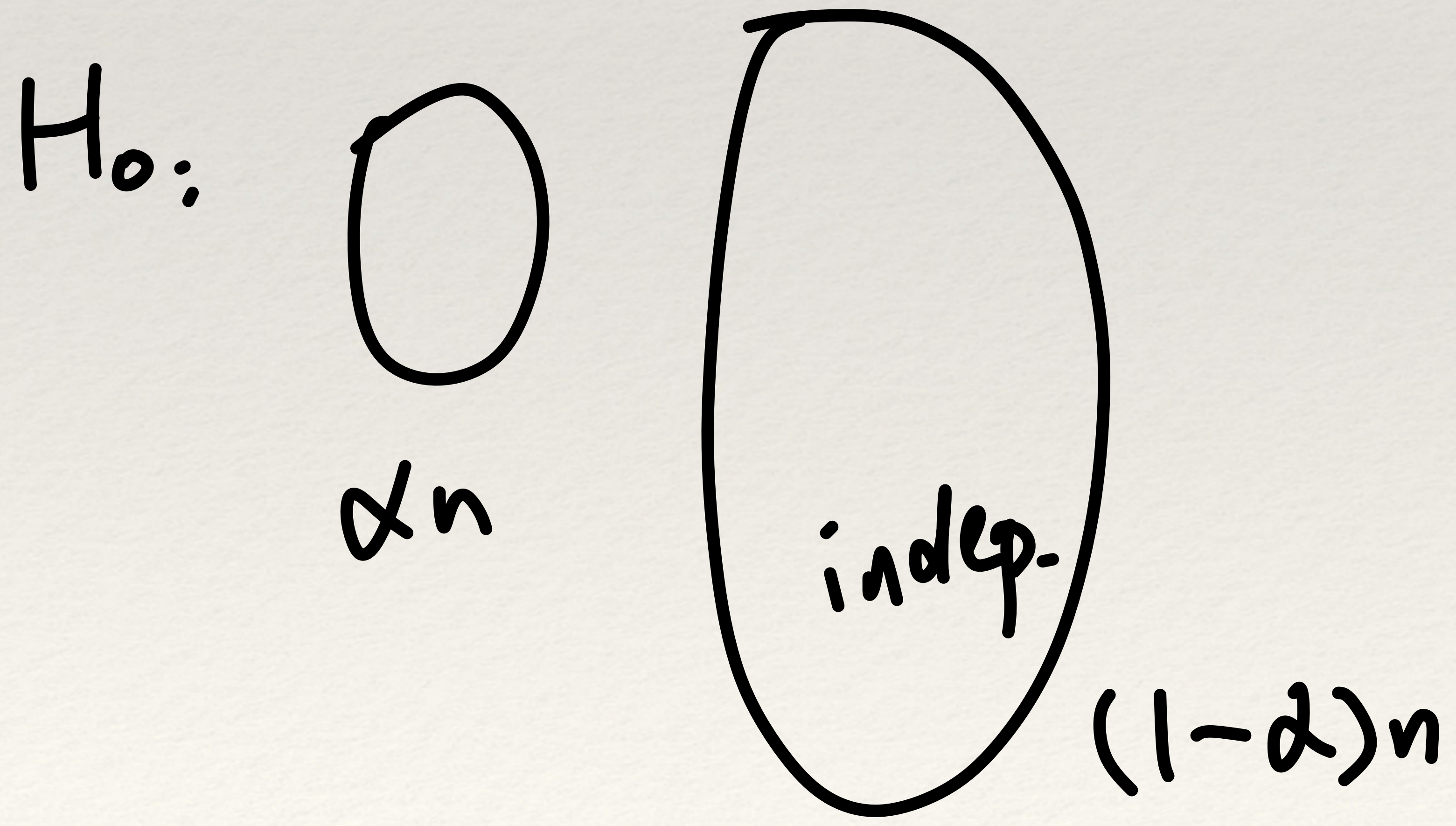
- First Goal: for $k \geq 3$, $\alpha > 0$,
  - $p = \omega(p_{k,r}) \Rightarrow \forall H$ with $\delta_{k-1}(H) \geq \alpha n$, whp. $H \cup H^{(k)}(n,p)$ has a K_r^k-factor, where $p_{k,r}$ is the threshold for almost K_r^k-factor in $H^{(k)}(n,p)$.
- A meta problem: for $k \geq 3$, $1 \leq d < k$, $\alpha > 0$,
  - $p = \omega(p_0) \Rightarrow \forall H$ with $\delta_d(H) \geq \alpha n^{k-d}$, whp. $H \cup H^{(k)}(n,p)$ has a spanning subgraph G, where p_0 is the threshold for the existence of an "almost spanning copy of G" in $H^{(k)}(n,p)$.
- Further Goal: work out the whole interval for $\alpha \in (0,1)$.

Perturbed k-graphs: clique-factors

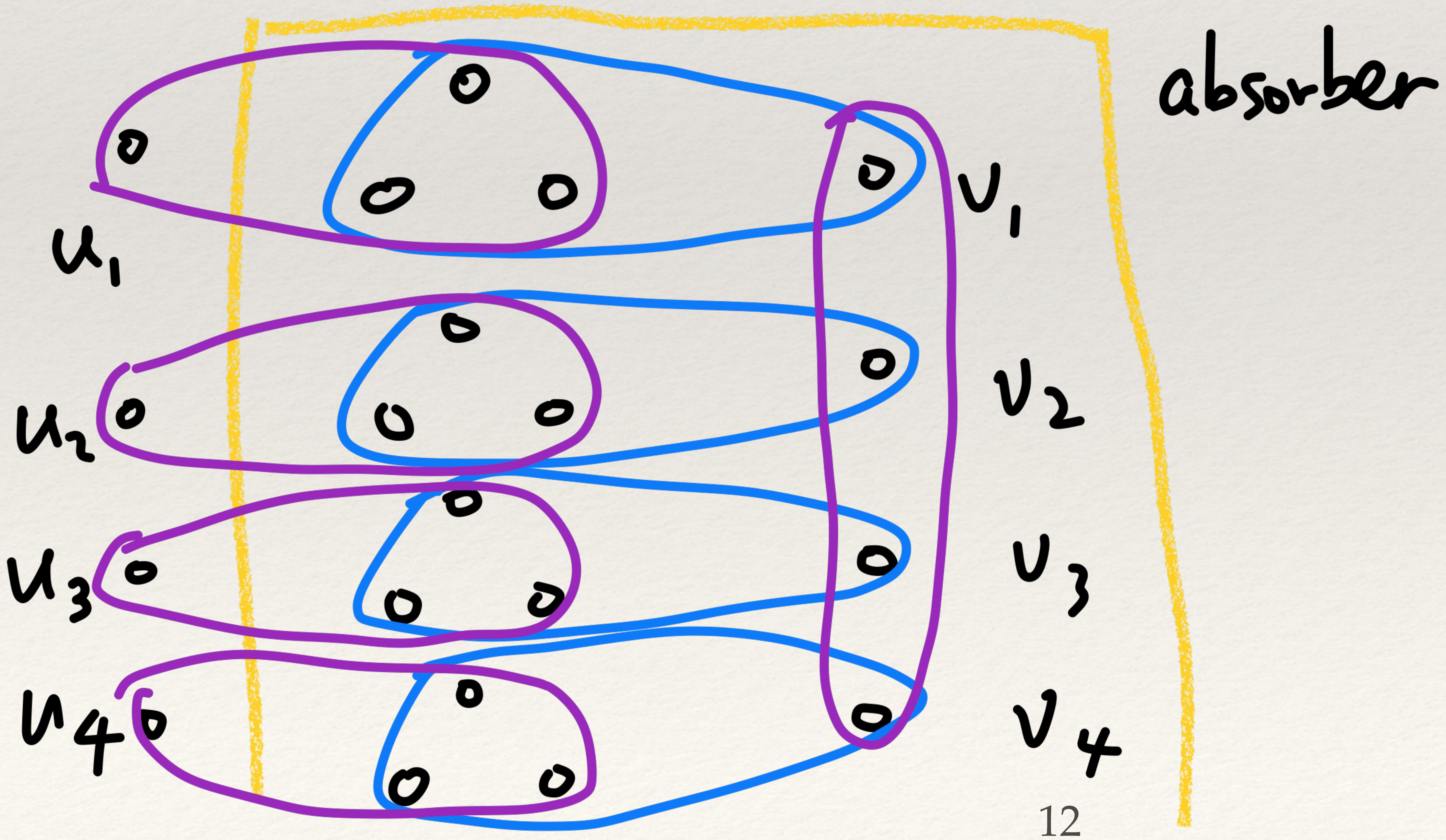
- (Chang-H.-Morris, ‘22++) for $k \geq 3$, $\alpha > 0$, $\exists r_0 = r_0(k)$, for $r \geq r_0$
  - $p = \omega(p_{k,r}) \Rightarrow \forall H$ with $\delta_{k-1}(H) \geq \alpha n$, whp. $H \cup H^{(k)}(n,p)$ has a K_r^k-factor, where $p_{k,r} = n^{(1-r)/\binom{r}{k}}$ is the threshold for almost K_r^k-factor in $H^{(k)}(n,p)$.
  - The case $k = 3$ is fully resolved: $r_0(3) = 4$
  - For $k \geq 4$, our proof fails for small cliques.

Proofs (for K_4^3). Lower bounds

- Let $H_0 := K_n^3 \setminus K_{(1-\alpha)n}^3$. Then a K_4^3-factor in $H_0 \cup H^{(3)}(n,p)$ needs $n/4 - \alpha n$ copies from $H^{(3)}(n,p)$.
    - Then the threshold p_0 should be the threshold for an almost K_4^3-factor ($p_0 = n^{-3/4}$ for this case).



Proofs (for K_4^3). Upper bounds

- Easy: by Janson's ineq, can find an almost K_4^3-factor in $H^{(3)}(n,p)$
- Absorption method: turn an almost K_4^3-factor to a perfect one
    - Absorber:
    - 

$u_1 \dots u_4$: to absorb

Each region represents a copy of K_4^3 (not an edge)

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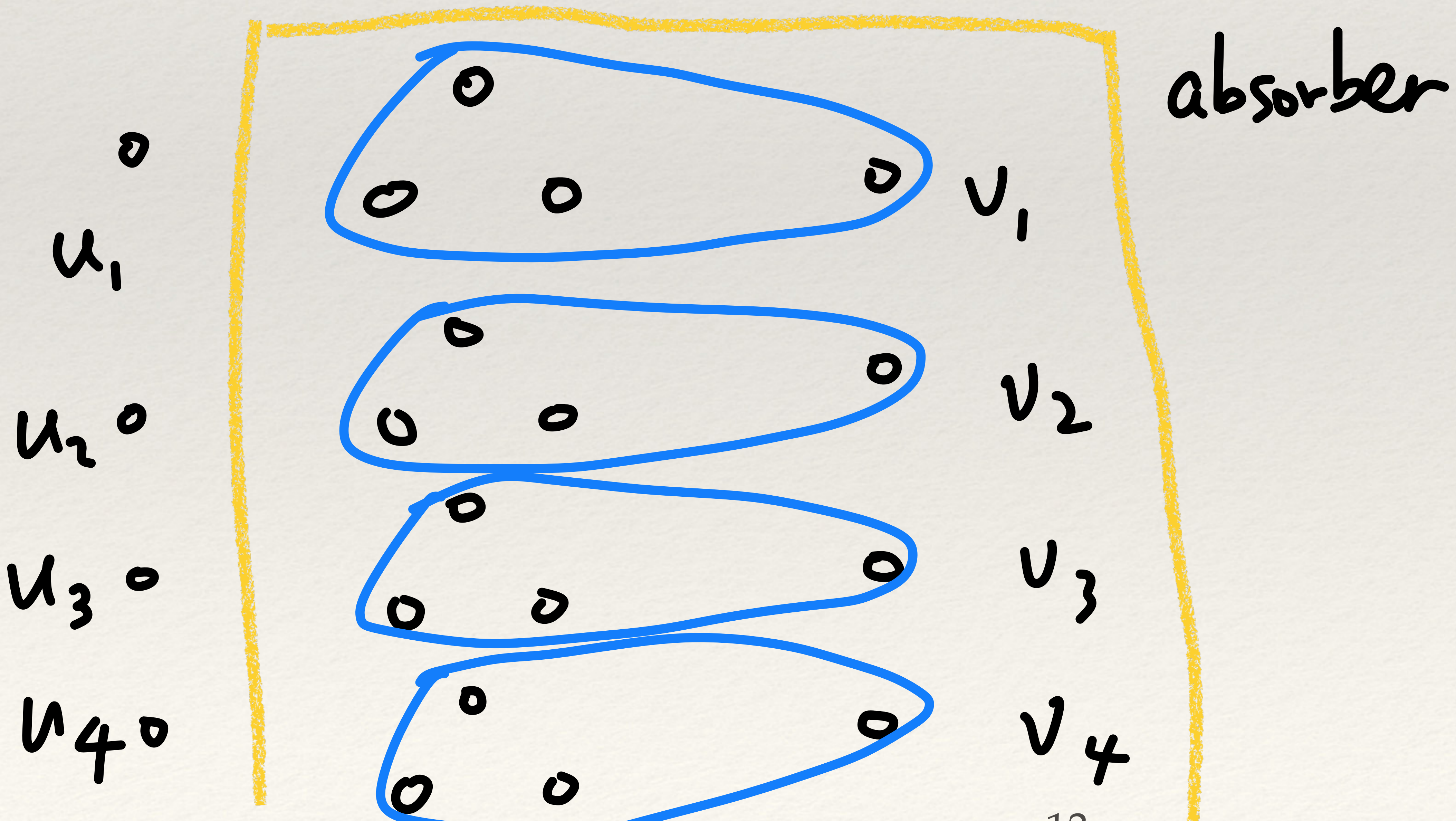

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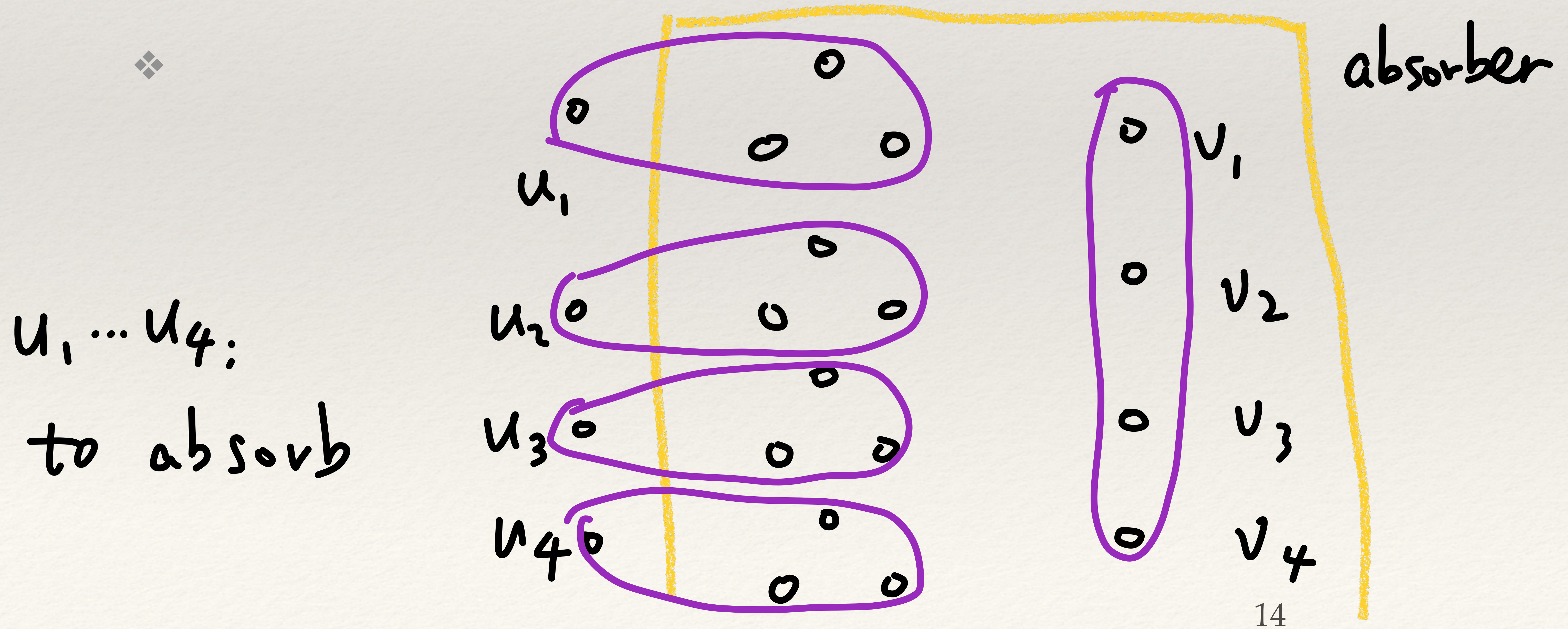
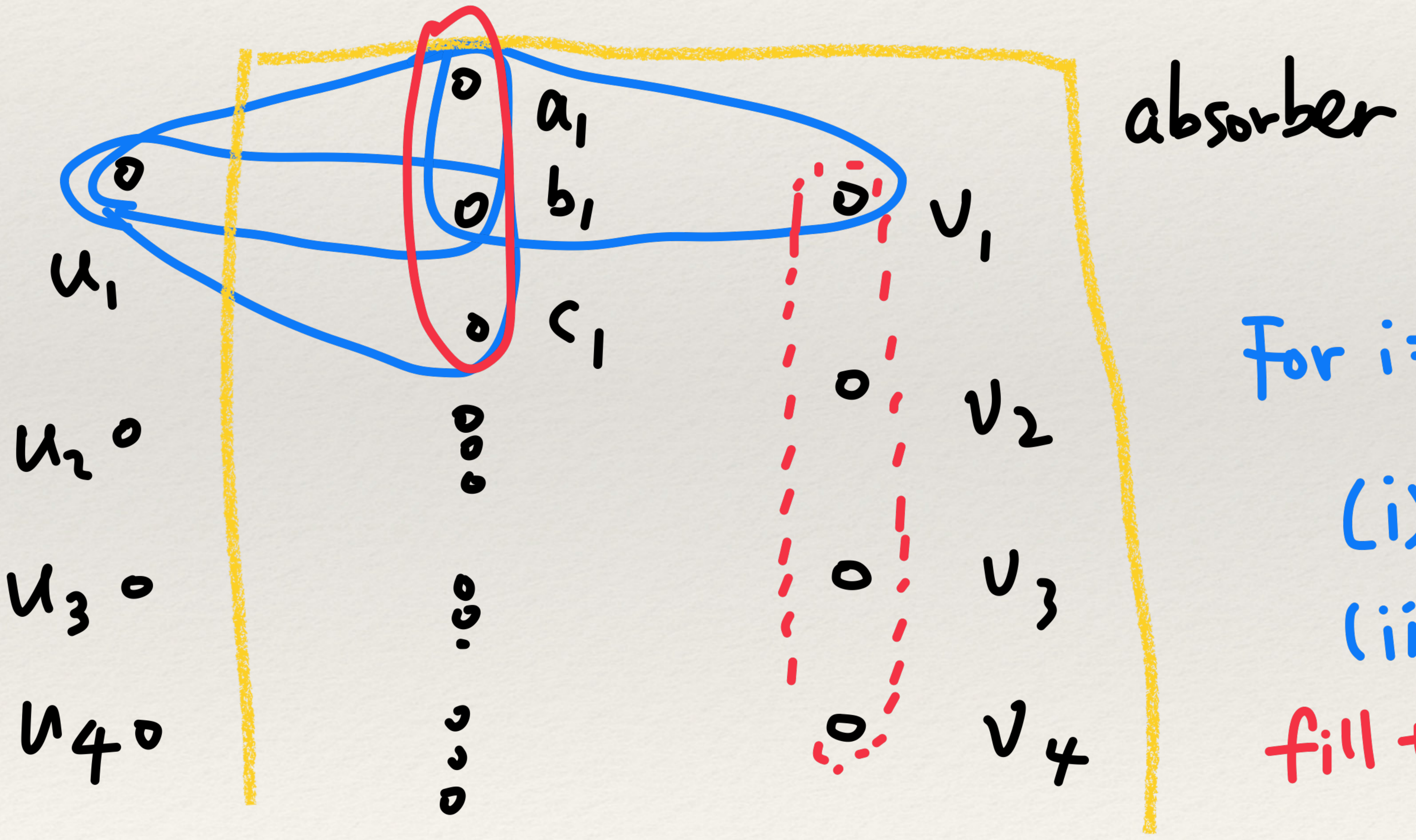


Proofs (for K_4^3). Upper bounds

- Construct an absorber for $\{u_1, u_2, u_3, u_4\}$:

$u_1 \cdots u_4$:
to absorb



blue: edges in H_0.

red: edges in $H^{(3)}(n,p)$.

For $i = 1, \cdots, 4$,

(i) Find $a_i b_i c_i$ as P_3 in $N(u_i)$,

(ii) Find $v_i \in N(a_i b_i)$

fill the missing edges by Janson's.

Proofs (for K_4^3). Construct absorbers

- Need to show, given a sequence of εn 4-tuples of vertices $\{Q_1, \ldots, Q_{\varepsilon n}\}$, find vtx-disjoint absorbers for them. (By bipartite-template, Montgomery, Kwan)
- If $\forall i$, Q_i has in expectation Cn absorbers, Janson's Ineq allows "greedily embedding": tail probability = $\exp(-Cn)$ and a union bound on $\varepsilon n \cdot 2^n$ possibilities for (i, W): in the i-th step, $W =$ unused vertices.
- A dedicated multi-round embedding scheme (found in a coffee shop in Valparaíso, Chile) only requires the expectation to be n^ε

An Embedding Scheme

- Given $\{Q_1, \ldots, Q_{\varepsilon n}\}$ s. t. $\forall i$, Q_i has in expectation $n^{0.1}$ absorbers.
- $\forall I \subseteq [\varepsilon n]$ of size $n^{0.9}$, $Q_i, i \in I$ in total have in expectation n absorbers.
- By Janson's ineq, can greedily find vtx-disjoint absorbers until there are $n^{0.9}$ Q_i's left.
- Now we are done the 1st round.

An Embedding Scheme - continued

- Take another copy of $H^{(3)}(n,p)$
- Given $\{Q_1, \ldots, Q_{n^{0.9}}\}$ s. t. $\forall i$, Q_i has in expectation $n^{0.1}$ absorbers.
- $\forall I \subsetneq [\varepsilon n]$ of size $n^{0.8}\log n$, Q_i, $i \in I$ in total have in expectation $n^{0.9}\log n$ absorbers in a vtx set W.
- By Janson's ineq, can greedily find vtx-disjoint absorbers until there are $n^{0.8}\log n$ Q_i's left.
  - In this round, need to consider $\binom{n^{0.9}}{n^{0.8}\log n} \le 2^{n^{0.8}\log^2 n}$ possibilities for I, and
  - $\binom{n}{n^{0.9}} \le 2^{n^{0.9}\log n}$ possibilities for the ground set W' for embedding.
- Now we are done the 2nd round, and it remains to deal with $n^{0.8}\log n$ Q_i's.

An Embedding Scheme - continued

- Repeat the argument for 11 rounds, with the number of leftover Q_i's being
- $n^{0.9}$, $n^{0.8}\log n$, $n^{0.7}\log^2 n$, …, $n^{0.1}\log^8 n$, $\log^9 n$, 0, and we are done.

Thanks for your attention.